The Furstenberg Structure Theorem in Topological Dynamics

Jan de Vries
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

In this paper I give a brief overview of the development of 'abstract' topological dynamics. Central in the research in this area after 1963 is Furstenberg's structure theorem for distal minimal flows. We give an almost completely self-contained proof of this theorem in its most general form (the 'relative', non-metric case). Every mathematician who feels at home in a compact Hausdorff space and who knows what a probability measure is should be able to understand the details. (For specialists: the proof does not use τ-topologies, Ellis groups or the circle operation.)

1 Introduction

Abstract topological dynamics deals with actions of groups on topological spaces: topological transformation groups (ttg's). More precisely, it is the study of ttg's with respect to those topological properties whose prototype occurred in classical dynamics (cf. [14], p.iii). Though in this paper this background is hardly recognizable, I will spend a few words to the development of the subject[1].

When at the end of the 19th century it became clear that for many important differential equations describing mechanical systems (in particular: those describing the solar system) it was extremely hard, or even impossible, to find explicit expressions for the solutions, Poincaré developed methods to discover significant features of those systems without integrating the equations. From his work and from later work by G.D. Birkhoff it followed that the study of many problems from this 'qualitative approach' could be performed in the framework of continuous actions of the group $\mathbb{R}$ on metric spaces. A standard reference for all major developments of the theory in this direction up to the middle 1940's is [18].

The abstract 'axiomatic' approach to the subject began with Gottschalk and Hedlund's book [14], where the topological study of dynamical systems was put in the framework of the action of an arbitrary topological group T (instead of $\mathbb{R}$) on an arbitrary topological Hausdorff space X (usually not metrizable): a continuous mapping π: $T \times X \longrightarrow X$ such that $\pi^e = id_X$ and $\pi^{st} = \pi^s \circ \pi^t$ for $s, t \in$

[1]) Many important developments in the theory of dynamical systems (e.g., ergodic theory, differential dynamics) will not be mentioned at all.

T (here e is the identity of T and $\pi^t : X \to X$ is defined by $\pi^t(x) := \pi(t, x)$ for $t \in T, x \in X$). In 1960 R. Ellis introduced the notion of the *enveloping semigroup* of a ttg $\langle T, X, \pi\rangle$ with X compact: the closure in X^X of the group $\{\pi^t : t \in T\}$ of homeomorphisms of X; this closure is a *semigroup* of (usually non-continuous) self-maps of X. It turned out that a number of important 'dynamical' notions are reflected in algebraic properties of this enveloping semigroup. Its role in 'abstract' topological dynamics can hardly be overestimated, but in this paper we shall pay no attention to it (at least, not explicitly). Since Ellis' work the emphasis of research in 'abstract' topological dynamics is on ttg's on compact spaces; such a compact ttg is called a *flow*. In particular, much attention has been given to the investigation of the structure and the classification problem of *minimal flows*, i.e., flows that have no proper closed invariant subsets. These problems are still far from being solved, and only certain subclasses of the class of all minimal flows have been studied.

The most simple class of minimal flows is formed by the equicontinuous ones. A flow $\langle T, X, \pi\rangle$ is called *equicontinuous* whenever $\{\pi^t : t \in T\}$ is an equicontinuous group of homeomorphisms of X; recall that in the case that X is a compact metric space with metric ρ this means:

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in X : \ \rho(x, y) < \delta \Rightarrow \rho(\pi^t x, \pi^t y) < \varepsilon \text{ for all } t \in T$$

(actually, this is *uniform* equicontinuity, which is equivalent with pointwise equicontinuity because X is compact).

If X is not metrizable then a definition can be given using the concept of a uniform structure: see Section 2. Examples of equicontinuous minimal flows can be obtained as follows. Consider any continuous homomorphism of topological groups $\phi : T \to G$, where G is a *compact* Hausdorff topological group, and assume that the image $\phi[T]$ of T in G under ϕ is dense in G. For any closed subgroup H of G, let X be the space of left cosets gH of H in $G\,(g \in G)$ endowed with the quotient topology. Then X is a compact Hausdorff space. An action of T on X can be defined in an obvious way by $\pi^t(gH) := \phi(t)gH$ $(t \in T, g \in G)$. In this way a ttg $\langle T, X, \pi\rangle$ is defined which is easily seen to be minimal and equicontinuous. It can be shown that *every equicontinuous minimal flow can be obtained in this way.* This result is the basis of a classification of equicontinuous minimal flows in terms of group compactifications of T.

It follows that in an equicontinuous minimal flow $\langle T, X, \pi\rangle$ the action of T actually is the restriction of an action of a compact group. In this case it is well known that if X is metrizable then there exists a metric on X, compatible with the topology of the space and invariant under the action of this compact group. In particular, this metric is invariant under the action of T, i.e., each π^t is an isometry. Thus, *a minimal flow $\langle T, X, \pi\rangle$ with X metric is equicontinuous iff it is isometric with respect to a compatible metric* ('only if' is outlined above, 'if' is obvious). We shall use this property as a motivation for two additional definitions: that of an isometric extension (see below) and that of a distal flow.

A flow $\langle T, X, \pi\rangle$ with X a compact metric space, say, with metric ρ, is *distal* whenever $\inf\{\rho(\pi^t x, \pi^t y) : t \in T\} > 0$ for every pair of distinct points $x, y \in X$. Obviously, if ρ is invariant under the action of T, i.e., if each π^t is an isometry, then $\langle T, X, \pi\rangle$ is distal. In particular, it follows from the above that *every* *equicontinuous minimal flow is distal* (actually, this is also true without the condition of minimality). When Ellis introduced distality in topological dynamics in 1958 it remained an open question whether the converse of the above statement holds: is every distal minimal flow equicontinuous? It was not before 1963 that H. Furstenberg discovered that Kakutani and Anzai's construction of skew product flows on the torus yield examples of minimal distal but not equicontinuous flows. (At the same time and independently another example was found in [2].) A simple example of this type is included at the end of this section. Furstenberg also proved a result that describes the relationship between distality and equicontinuity for minimal flows, now known as Furstenberg's Structure Theorem (FST). In order to state this result we need an additional definition, viz., the notion of a minimal flow being *isometric 'over' another flow* (or: isometric *'relative'* another flow); this notion generalizes the concept of a minimal flow with an invariant metric which, as we have seen above, is the same as an equicontinuous minimal flow on a metric space (hence we shall also speak over minimal flows *equicontinuous* over another flow).

Suppose we have two flows $\langle T, X, \pi\rangle$ and $\langle T, Y, \sigma\rangle$ and a continuous surjection $\phi : X \rightarrow Y$ such that $\phi \circ \pi^t = \sigma^t \circ \phi$ for every $t \in T$. Then ϕ, but also $\langle T, X, \pi\rangle$, is called an *extension* of $\langle T, Y, \sigma\rangle$; notation: $\phi : \langle T, X, \pi\rangle \rightarrow \langle T, Y, \sigma\rangle$. An extension $\phi : \langle T, X, \pi\rangle \rightarrow \langle T, Y, \sigma\rangle$ is called *isometric* whenever there exists a continuous mapping $\rho : X \times X \rightarrow \mathbb{R}$ which induces a compactible *metric on each fiber* $\phi^{\leftarrow}[y]$ ($y \in Y$; thus, for example, the triangle inequality $\rho(x_1, x_3) \leq \rho(x_1, x_2)+\rho(x_3, x_2)$ is required only for triples x_1, x_2, x_3 in X with $\phi(x_1) = \phi(x_2) = \phi(x_3)$) and such that each π^t induces an isometry of $\phi^{\leftarrow}[y]$ onto $\pi^t\phi^{\leftarrow}[y] = \phi^{\leftarrow}[ty]$ ($t \in T, y \in Y$; so $\rho(\pi^t x_1, \pi^t x_2) = \rho(x_1, x_2)$ for all pairs of points x_1, x_2 in X with $\phi(x_1) = \phi(x_2)$). Note that ρ is not required to be a metric on all of X. If $\phi : \langle T, X, \pi\rangle \rightarrow \langle T, Y, \sigma\rangle$ is an isometric extension then the flow $\langle T, X, \pi\rangle$ is called isometric over $\langle T, Y, \sigma\rangle$. Isometric extensions are well-analyzed; there is a close connection with fibre bundles (see [9], p. 481, 482 and [3], 3.17.4)

It is rather easy to see that an isometric extension of a distal flow is distal ⟦briefly: if $x_1, x_2 \in X$ are in the same fiber then they keep the same distance under the action of T because the extension is isometric; if x_1, x_2 are not in the same fiber then they cannot approach each other under the action of T because their images in the distal flow cannot approach each other ⟧. So if we start with an isometric flow and if we cansider flows that can be obtained by 'towers' of successive isometric extensions (possibly infinitely often: then one has to take inverse limits, a process that also turns out to preserve distality) then we obtain again distal flow. In 1963 H. Furstenberg was able to show that *every metric distal minimal flow can be obtained in this way.*

This result of Furstenberg's (the FST: Furstenberg's Structure Theorem) had an enormous impact on abstract topological dynamics. First of all, it caused a shift in attention from minimal flows to extensions of minimal flows. In addition, people tried to obtain similar results for other types of flows and extensions (e.g., with 'distal' weakened to 'point-distal'). In this paper we shall concentrate on the following aspects: in the FST metrizability of X can be omitted (see [7]), and the result not only applies to distal minimal flows but also to *distal extensions* of minimal flows (see [5], 15.4 for the metric - more generally: 'quasi-separable'

- case, and [17] for the general case). In the proofs of these generalizations the original idea of Furstenberg, viz., the introduction of auxiliary topologies (often also called τ-topologies: see e.g. [8] or [13]), was extended and refined. In the remainder of this paper a different approach to the general FST will be sketched; though this approach seems to be new, it is based on techniques and results that have been known for some 10 years. Perhaps this approach is not much 'easier' than the approach based on τ-topologies, but it has the advantage that it uses only a small number of main steps, each of which has its own interest.

EXAMPLE. (See [9]; for minimality: p.36/37 in [10].) Let $T = \mathbb{Z}$, $X = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ and let $\pi : T \times X \to X$ be defined by $\pi(n, x) := f^n(x)$ $(n \in \mathbb{Z}, x \in X)$, where f is the homeomorphism of X defined by

$$f(x_1, x_2) := (x_1 + \alpha, x_2 + x_1) \text{ for } x = (x_1, x_2) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}).$$

Here '+' means addition mod 1, and α is an irrational number.

It is easy to see that this flow is distal: if (x_1, x_2) and (y_1, y_2) are points of $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ and $x_1 \neq y_1$ then the distance of the points $f^n(x_1, x_2)$ and $f^n(y_1, y_2)$ is at least the difference of their first coordinates, which is independent of n; and if $x_1 = y_1$ then $f^n(x_1, x_2)$ and $f^n(y_1, y_2)$ have equal first coordinates but the difference of their second coordinates is independent of n. It is also an easy exercise to show that this flow is not equicontinuous: if $\{f^n : n \in \mathbb{Z}\}$ were (uniformly) equicontinuous then for every sequence $(x_1^{(n)}, x_2^{(n)}) \rightsquigarrow (0,0)$ the sequence $\{f^n(x_1^{(n)}, x_2^{(n)})\}_{n \in \mathbb{N}}$ would tend to (0,0) for $n \rightsquigarrow \infty$; but taking $x_1^{(n)} = (4n)^{-1}$ and $x_2^{(n)} = 0$, we see that this is not the case. It takes more effort to show that this flow is minimal. (We do not need the argument in the remainder of this paper, but some of the techniques are so widely used in abstract topological dynamics that we include the proof.) Since X is compact, a straightforward Zorn-argument shows that X has a *minimal set* F (i.e., F is invariant, that is, $f^n[F] = F$ for all $n \in \mathbb{Z}$, $F \neq \emptyset$ and F is minimal (under inclusion) for these properties). We want to show that $F = X$.

To this end, define an action of the group $\mathbb{R}/\mathbb{Z}$ on X by $(g, (x_1, x_2)) \mapsto g \cdot (x_1, x_2) := (x_1, x_2 + g) : (\mathbb{R}/\mathbb{Z}) \times X \to X$. It is clear that each $g \in \mathbb{R}/\mathbb{Z}$ acts as a homeomorphism of X which commutes with f (an *automorphism* of the flow $\langle T, X, \pi \rangle$). It follows easily that $g \cdot F$ is a minimal set for every $g \in \mathbb{R}/\mathbb{Z}$ $[\![$if F_1 is closed and invariant then so is $g \cdot F_1]\!]$, hence $g \cdot F \cap F$, being a closed invariant set in each of the minimal sets F and $g \cdot F$, is either empty or equal to $g \cdot F = F$. Clearly, $H := \{g \in \mathbb{R}/\mathbb{Z} : g \cdot F = F\}$ is a subgroup of $\mathbb{R}/\mathbb{Z}$, and using that for $g \in \mathbb{R}/\mathbb{Z}$ we have $g \notin H$ iff $g \cdot F \subseteq X \setminus F$ it is not to difficult to show that $(\mathbb{R}/\mathbb{Z}) \setminus H$ is open $[\![$compactness argument$]\!]$. So H is a closed subgroup of $\mathbb{R}/\mathbb{Z}$. It is sufficient to show that $H = \mathbb{R}/\mathbb{Z}$. Indeed, in that case $F = X = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, as follows. Note that the projection $\phi : X \to \mathbb{R}/\mathbb{Z}$ onto the *first* coordinate maps F onto a closed set in $\mathbb{R}/\mathbb{Z}$ which is invariant under the homeomorphism ('rotation') $x \mapsto x + \alpha$. As $\mathbb{R}/\mathbb{Z}$ is minimal under this homeomorphism $[\![\alpha \notin \mathbb{Q}$, so every 'orbit' $\{x + n\alpha \pmod 1 : n \in \mathbb{Z}\}$ is dense in $\mathbb{R}/\mathbb{Z}]\!]$ it follows that $\phi[F] = \mathbb{R}/\mathbb{Z}$. Hence for every $(x_1, x_2) \in X$ there is a point of the form (x_1, x_2') in F. But then for suitable $g \in \mathbb{R}/\mathbb{Z}$ we have

$(x_1, x_2) = g \cdot (x_1, x_2') \in g \cdot F$. If $\mathbb{R}/\mathbb{Z} = H$ then $g \cdot F = F$, hence $(x_1, x_2) \in F$. So in this case $X = F$, and X is minimal under f.

Assume that $H \neq \mathbb{R}/\mathbb{Z}$; then there exists a character (i.e., a continuous homomorphism of groups) $\chi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ with $\chi \neq 0$ and $\chi\,|_H = 0$ (see [15], 23.26). For every $x \in \mathbb{R}/\mathbb{Z}$ there exists $g \in \mathbb{R}/\mathbb{Z}$ with $(x, g) \in F$ ⟦see above⟧, i.e., $g \cdot (x, 0) \in F$; put $\tau(x) := \chi(g)$. In this way one unambiguously defines a mapping $\tau : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ ⟦if also $(x, g_1) \in F$ then $(-g + g_1) \cdot F \cap F \neq \emptyset$, so $g_1 - g \in H$ and $\chi(g_1) = \chi(g)$⟧. In order to show that τ is continuous, note that the graph of τ is the following subset of $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$:

$$\Gamma(\tau) = \{(x, \tau(x)) : x \in \mathbb{R}/\mathbb{Z}\} = \{(x, \chi(g)) : (x, g) \in F\} = (id_{\mathbb{R}/\mathbb{Z}} \times \chi)[F].$$

As F is compact and $id_{\mathbb{R}/\mathbb{Z}} \times \chi$ is continuous it follows that $\Gamma(\tau)$ is closed; as $\mathbb{R}/\mathbb{Z}$ is compact it follows that τ is continuous ([21], Problem 119 in Sec. 6.7). Now note that for $(x, g) \in F$ also $(x + \alpha, g + x) = f(x, g) \in F$, so by the definition of τ:

$$\tau(x + \alpha) = \chi(g + x) = \chi(g) + \chi(x) = \tau(x) + \chi(x).$$

However, the character χ on $\mathbb{R}/\mathbb{Z}$ is of the form $x \mapsto nx$ for some $n \in \mathbb{Z}$, $n \neq 0$ because $\chi \neq 0$ on $\mathbb{R}/\mathbb{Z}$ (see [15], 23.27(a)). So we get for every $x \in \mathbb{R}/\mathbb{Z}$: $\tau(x + \alpha) = \tau(x) + nx$. Now a homotopy argument shows that this is impossible. Heuristically this argument is as follows: view $\mathbb{R}/\mathbb{Z}$ as the unit circle in $\mathbb{C}$; if x runs precisely once counter-clockwise through this circle then so does $x + \alpha$, but according to the above formula $\tau(x+\alpha)$ makes n more complete windings around the circle than $\tau(x)$ does, which is impossible if τ is continuous and $n \neq 0$. So $H \neq \mathbb{R}/\mathbb{Z}$ is impossible. □

This example also illustrates Furstenberg's Structure Theorem. For let $\langle \mathbb{Z}, Y, \sigma \rangle$ be the flow with $Y = \mathbb{R}/\mathbb{Z}$ and $\sigma(n, y) := h^n(y)$ for $(n, y) \in \mathbb{Z} \times Y$, where h is the homeomorphism of Y defined by $h(y) = y + \alpha$ $(y \in Y)$. As was already observed above, $\langle \mathbb{Z}, Y, \sigma \rangle$ is a minimal flow. Moreover, the continuous surjection $\phi : (x_1, x_2) \mapsto x_1 : X \to Y$ satisfies the equality $\phi \circ f = h \circ \phi$, hence $\phi \circ f^n = h^n \circ \phi$ for all $n \in \mathbb{Z}$. So $\phi : \langle \mathbb{Z}, X, \pi \rangle \to \langle \mathbb{Z}, Y, \sigma \rangle$ is an *extension of minimal flows*. It has the following properties:

(a) On every fiber $\{x\} \times (\mathbb{R}/\mathbb{Z})$ of ϕ the group $\{f^n : n \in \mathbb{Z}\}$ acts as a family of isometries, each mapping this fiber onto the fiber $\{h(x)\} \times \mathbb{R}/\mathbb{Z}$.

(b) The flow $\langle \mathbb{Z}, Y, \sigma \rangle$ is isometric.

Thus, $\langle \mathbb{Z}, X, \pi \rangle$ is an isometric extension of an isometric flow, in accordance with FST.

2 Notation and preliminaries

In the remainder of this paper, T will be an arbitrary topological group with identity e; the group operation in T will be written as a multiplication[2]. A T-*flow*, or just: a *flow* $\mathcal{X}$ consists of a compact Hausdorff space X (the *phase space*

[2]) Much of what follows is trivial if T is compact, so T should be assumed to be not compact. In fact, the topology of T will play no explicit role at all, so T might be assumed to be discrete.

of $\mathcal{X}$) and a continuous action of T on X. We shall not use special symbols to denote actions; thus, a (continuous) action of T on X is a (continuous) mapping $(t, x) \mapsto tx : T \times X \to X$ such that $s(tx) = (st)x$ and $ex = x$ for $s, t \in T$ and $x \in X$. Flows will be denoted by script symbols $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots)$ and their phase spaces by the corresponding capitals in italics $(X, Y, Z, \ldots)$. A *morphism of flows* $\phi : \mathcal{X} \to \mathcal{Y}$ is a continuous mapping $\phi : X \to Y$ such that $\phi(tx) = t\phi(x)$ for all $t \in T$ and $x \in X$. A morphism is called injective, surjective or bijective whenever the underlying continuous mapping is injective, surjective or bijective, respectively. A surjective morphism $\phi : \mathcal{X} \to \mathcal{Y}$ is called an *extension of* $\mathcal{Y}$ (in that case $\mathcal{Y}$ is called a *factor of* $\mathcal{X}$), and a bijective morphism $\phi : \mathcal{X} \to \mathcal{Y}$ is called an *isomorphism*; in the latter case, $\phi : X \to Y$ is a homeomorphism and $\phi^{-1} : \mathcal{Y} \to \mathcal{X}$ is also an isomorphism. If $\xi : \mathcal{X} \to \mathcal{Z}$ and $\eta : \mathcal{Y} \to \mathcal{Z}$ are extensions then we shall say that η *is a factor of* ξ (relative $\mathcal{Z}$) whenever there exists a surjective morphism $\phi : \mathcal{X} \to \mathcal{Y}$ such that $\xi = \eta \circ \phi$ (such a ϕ is not necessarily unique). Note that a flow $\mathcal{Y}$ is a factor of a flow $\mathcal{X}$ iff the extension $\mathcal{Y} \to (*)$ is a factor of the extension $\mathcal{X} \to (*)$; here $(*)$ denotes the trivial flow consisting of one point (with the obvious action of T).

A subset A of the phase space of a flow $\mathcal{X}$ is *invariant* whenever $TA = A$, i.e., whenever $Tx \subseteq A$ for all $x \in A$ (here $Tx := \{tx : t \in T\}$ is the *orbit* of x); A is called a *minimal set* in $\mathcal{X}$ whenever $A \neq \emptyset$, A is closed and invariant and A is minimal (under inclusion) for these properties, i.e., if $\emptyset \neq B \subseteq A$, B closed and invariant, then $B = A$. Clearly, a non-empty subset A of X is minimal iff $\overline{Tx} = A$ for every $x \in A$ (the bar denotes closure; note that the closure of an invariant set is invariant). If X is a minimal set in $\mathcal{X}$ then $\mathcal{X}$ is called a *minimal flow*. If $\phi : \mathcal{X} \to \mathcal{Y}$ is a morphism of flows and A is a (closed) invariant set in $\mathcal{X}$ then $\phi[A]$ is a (closed) invariant set in $\mathcal{Y}$. So if $\mathcal{Y}$ is minimal then $\phi[A] = Y$; in particular, ϕ is surjective. If B is a (closed) invariant set in $\mathcal{Y}$ then $\phi^{\leftarrow}[B]$ is a (closed) invariant set in $\mathcal{X}$. So if A is a minimal set in $\mathcal{X}$ then $\phi[A]$ is minimal in $\mathcal{Y}$ $[\![$consider a closed invariant set $B \subseteq \phi[A]$: then $A \cap \phi^{\leftarrow}[B] \neq \emptyset$ hence $A \subseteq \phi^{\leftarrow}[B]$, and $\phi[A] = B]\!]$; in particular, if $\phi : \mathcal{X} \to \mathcal{Y}$ is an extension and $\mathcal{X}$ is minimal then $\mathcal{Y}$ is minimal.

Recall that a compact space X has a unique uniform structure U_X compatible with the topology: all subsets α of $X \times X$ that include the diagonal $\Delta_X := \{(x, x) : x \in X\}$ in their interior (with respect to the product topology on $X \times X$). A flow $\mathcal{X}$ is *equicontinuous* whenever T acts as a (uniformly) equicontinuous group of transformations of X with respect to the uniform structure U_X, i.e.,

$$\forall \alpha \in U_X \ \exists \beta \in U_X \ : \ T\beta \subseteq \alpha.$$

Here $T\beta$ has to be interpreted in the flow $\mathcal{X} \times \mathcal{X}$, i.e., the coordinate-wise action of T on $X \times X$; thus, $T\beta := \{(tx_1, tx_2) : t \in T \,\&\, (x_1, x_2) \in \beta\}$. (The equicontinuous minimal flows are well understood. They can be classified by means of (classes of non-conjugated) closed subgroups of the Bohr compactification of T; see e.g. [1], Theorem 6 on p. 53. Briefly, this is because their enveloping semigroups are compact topological groups, hence factors of the Bohr compactification.)

A flow $\mathcal{X}$ is *distal* whenever for every pair of *distinct* points $x_1, x_2 \in X$ the closure of the orbit of (x_1, x_2) in the flow $\mathcal{X} \times \mathcal{X}$ is disjoint from Δ_X, i.e., $\overline{T(x_1, x_2)} \cap \Delta_X = \emptyset$; equivalently: if $x_1 \neq x_2$ then there is $\alpha \in U_X$ such that

$T(x_1, x_2) \cap \alpha = \emptyset$, which can be expressed by saying that tx_1 and tx_2 remain always α-apart from each other (α depends on (x_1, x_2)). It is easily seen that in case X is metrizable this definition is equivalent with the one in the Introduction. A fundamental result is that *the phase space of a distal flow is a union of minimal sets.* (This is not a trivial result. It follows quite easily from the fact that for a distal flow the enveloping semigroup is a group. In this context the following terminology is convenient. In an arbitrary flow $\mathcal{X}$ a pair of points $x_1, x_2 \in X$ is called a *proximal* pair whenever $\overline{T(x_1, x_2)} \cap \Delta_X \neq \emptyset$, i.e., whenever there is a net $\{t_\gamma\}_{\gamma \in \Gamma}$ in T and there is a point x in X such that $t_\gamma x_i \rightsquigarrow x$ along Γ for $i = 1, 2$. If $x_1, x_2 \in X$ and either $x_1 = x_2$ or (x_1, x_2) is not a proximal pair then (x_1, x_2) is called a *distal* pair. Thus, (x_1, x_2) is both a distal and a proximal pair iff $x_1 = x_2$. Clearly, a flow $\mathcal{X}$ is distal iff all pairs of points $x_1, x_2 \in X$ form a distal pair. Similarly, a flow $\mathcal{X}$ is called *proximal* whenever all pairs of points $x_1, x_2 \in X$ are proximal. (It can be shown that if T is Abelian, or more generally: $T = KS$ with K a compact set in T and S a nilpotent normal subgroup, then T admits no non-trivial proximal minimal flows; cf. [13], II.3.4. The natural action of $T := SL(2, \mathbb{R})$ on the projective line $\mathbb{P}^1$ - i.e., the unit circle in $\mathbb{R}^2$ with antipodal points identified - is a proximal minimal flow: $\mathbb{P}^1$ is even homogeneous under $SL(2, \mathbb{R})$, and proximality follows from the fact that under the sequence $\{t_k\}_{k \in \mathbb{N}}$ with $t_k = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ all points of $\mathbb{P}^1$ converge to the same point.)

It is not difficult to show that *equicontinuous* $\Rightarrow$ *distal* for any flow $\mathcal{X}$. ⟦If $x_1, x_2 \in X$ and $x_1 \neq x_2$ then $(x_1, x_2) \notin \alpha$ for some $\alpha \in U_X$; let $\beta \in U_X$ with $T\beta \subseteq \alpha$. Then $T(x_1, x_2) \cap \beta = \emptyset$, for otherwise $(x_1, x_2) \in T\beta \subseteq \alpha$.⟧ The example in the Introduction shows that the converse is not true in general, not even for minimal flows. The notions 'proximal' and 'distal' are opposed to each other: if a flow $\mathcal{X}$ is both proximal and distal then $\mathcal{X} = (*)$ ⟦now $X \times X = \Delta_X$⟧. (Actually, more can be said: *if $\mathcal{X}$ is a distal minimal flow and $\mathcal{Y}$ is a proximal minimal flow then $\mathcal{X} \times \mathcal{Y}$ is minimal*: cf. [13], II.1.3. Much attention has been devoted to the question for which classes $\mathbf{M}_1$ and $\mathbf{M}_2$ of minimal flows one has $\mathcal{X} \times \mathcal{Y}$ minimal for $\mathcal{X} \in \mathbf{M}_1$ and $\mathcal{Y} \in \mathbf{M}_2$; see e.g. Chapter VI in [22].)

A flow $\mathcal{X}$ is said to be *ergodic* whenever each invariant set is either dense or nowhere dense (this is adapted from ergodic theory, where a measure preserving flow is called ergodic whenever every invariant measurable set has measure 0 or has a complement of measure 0); equivalently, $\mathcal{X}$ is ergodic iff every non-empty open invariant set is dense, iff for every pair of non-empty open sets U and V in X one has $TU \cap V \neq \emptyset$. *If X is metrizable then $\mathcal{X}$ is ergodic iff there is a point in X with dense orbit.* ⟦'If' : Let $\overline{Tx} = X$ and let $\emptyset \neq U \subseteq X$, U open and invariant. Then $U \cap Tx \neq \emptyset$, hence $x \in TU = U$ and $Tx \subseteq U$, so U is dense. 'Only if': Let $\mathcal{B}$ be a countable base for the topology of X, and assume $\emptyset \notin \mathcal{B}$. For every $U \in \mathcal{B}$, $TU = \bigcup\{tU : t \in T\}$ is an union of open sets (each $t \in T$ acts as a homeomorphism of X) so TU is open; as TU is invariant it is dense (ergodicity). By Baire's theorem, $D := \bigcap\{TU : \in \mathcal{B}\}$ is not empty. For $x \in D$ the orbit Tx meets every $U \in \mathcal{B}$, hence is dense.⟧

A flow $\mathcal{X}$ is called *weakly mixing* whenever the flow $\mathcal{X} \times \mathcal{X}$ is ergodic. This is the opposite of equicontinuity: *if $\mathcal{X}$ is both equicontinuous and weakly mixing then $\mathcal{X} = (*)$.* ⟦Suppose there are $x_1, x_2 \in X$, $x_1 \neq x_2$. Then there is $\alpha \in U_X$

with $(x_1, x_2) \notin \overline{\alpha}$. Let $\beta \in U_X$ be such that $T\beta \subseteq \alpha$ (equicontinuity). As β has non-empty interior in $X \times X$ we have $\overline{T\beta} = X \times X$ ($\mathcal{X} \times \mathcal{X}$ is ergodic), hence $\overline{\alpha} = X \times X$. Contradiction.⟧ In this statement 'equicontinuous' can be replaced by the weaker property 'distal'; this is a consequence of the following theorem: *if a flow $\mathcal{X}$ is both distal and ergodic then $\mathcal{X}$ is minimal.* ⟦Indeed, if $\mathcal{X}$ is distal and weakly mixing then $\mathcal{X} \times \mathcal{X}$ is easily seen to be distal; as $\mathcal{X} \times \mathcal{X}$ is ergodic, the result implies that $\mathcal{X} \times \mathcal{X}$ is minimal. But Δ_X is closed and invariant, so $X \times X = \Delta_X$, hence $\mathcal{X} = (*)$.⟧ The proof of this theorem is rather easy in the case that X is metrizable: in that case ergodicity of $\mathcal{X}$ implies that $\mathcal{X}$ has a point with dense orbit (see above). Distality of $\mathcal{X}$ implies that $\mathcal{X}$ is a union of minimal sets. So if $\mathcal{X}$ is both distal and ergodic then $\mathcal{X}$ is minimal. In the case that X is not metrizable the result was proved in [7] by an ingenious reduction to the metric case (see 4.1.1 below).

It is easy to see that a weakly mixing flow has no non-trivial equicontinuous factor ⟦every factor of a weakly mixing flow is weakly mixing, hence if it is equicontinuous then it is trivial⟧. Rather simple examples show that the converse need not be true, not even for minimal flows. Much attention has been paid to the question under which additional conditions a minimal flow without non-trivial equicontinuous factors is weakly mixing; see [22], Chap. VII for an overview. This question is related to the FST; see below.

The above properties for flows are examples of 'absolute' properties. Flows can also have properties 'relative a factor'; in practice those 'relative' properties are properties of extensions such that, when applied to extensions of the form $\mathcal{X} \to (*)$ one gets the corresponding 'absolute' property for $\mathcal{X}$[3]. 'Relativization' of the above mentioned properties yields the following:

An extension $\phi : \mathcal{X} \to \mathcal{Z}$ is called *proximal* or *distal* whenever all pairs of points $x_1, x_2 \in X$ for which $\phi(x_1) = \phi(x_2)$ are proximal or distal, respectively. Clearly, a flow $\mathcal{X}$ is proximal or distal iff $\mathcal{X} \to (*)$ is a proximal or distal extension. In the same vain, an extension $\phi : \mathcal{X} \to \mathcal{Z}$ is *equicontinuous* whenever the equicontinuity condition for $\mathcal{X}$ holds only on fibers of ϕ (and uniformly so in the fibers):

$$\forall \alpha \in U_X \ \exists \beta \in U_X : \quad (x_1, x_2) \in \beta \ \& \ \phi(x_1) = \phi(x_2) \Rightarrow (tx_1, tx_2) \in \alpha \text{ for all } t \in T.$$

(Equicontinuous extensions have a well-understood structure. For example, it can be shown that if $\phi : \mathcal{X} \to \mathcal{Z}$ is an equicontinuous extension of minimal flows then $\phi : X \to Z$ is a fibre bundle: there is an extension $\theta : \mathcal{X}_0 \to \mathcal{Z}$ of which ϕ is a factor, say under $\psi : \mathcal{X}_0 \to \mathcal{X}$ (i.e., $\theta = \phi \circ \psi$) which is of the following special form: there is a compact group K acting continuously as a group of automorphisms of $\mathcal{X}_0$ such that the K-orbits in $\mathcal{X}_0$ are just the fibers of ψ. It follows that every fiber $\phi^{\leftarrow}[z]$ of ϕ is homeomorphic with K/K_0 for some closed subgroup K_0 of K ($z \in Z$). For details, see [3], 3.17.4, combined with [8], 1.13. This can be used to show that if, in addition, X is metrizable, then ϕ

[3]) Extensions can also have properties that are not 'relative' in this sense, i.e., that have no meaningful 'absolute' counterpart (e.g. because $\mathcal{X} \to (*)$ has such a property iff $\mathcal{X} = (*)$, or because $\mathcal{X} \to (*)$ *always* has the property; an example of the latter is the property for extensions of being an open mapping).

is isometric according to the definition in the Introduction: see [5], p. 159 (this proof is not correct; for another proof, see 4.3.5 below). This structure can easily be recognized in the example in the Introduction.) As in the absolute case one shows that every equicontinuous extension is distal ⟦insert '$\phi(x_1) = \phi(x_2)$' in the appropriate places in the proof of the absolute case⟧, and that an extension which is both distal and proximal is 'trivial' in the sense that it is an isomorphism ⟦every fiber consists of one point, i.e., ϕ is injective, hence a homeomorphism⟧.

For an extension $\phi : \mathcal{X} \to \mathcal{Z}$ the set

$$R_\phi := \{(x_1, x_2) \in X \times X \ : \ \phi(x_1) = \phi(x_2)\}$$

(sometimes it is convenient to write $R(\phi)$ for R_ϕ) is a non-empty ⟦$\Delta_X \subseteq R_\phi$⟧ closed ⟦$\phi$ is continuous⟧ invariant ⟦$\phi(x_1) = \phi(x_2)$ implies $\phi(tx_1) = t\phi(x_1) = t\phi(x_2) = \phi(tx_2)$⟧ set in the flow $\mathcal{X} \times \mathcal{X}$. By restriction of transformations it defines a 'subflow' of $\mathcal{X} \times \mathcal{X}$, denoted by $\mathcal{R}_\phi$. Note that an extension $\phi : \mathcal{X} \to \mathcal{Y}$ is completely determined by R_ϕ: if also $\psi : \mathcal{X} \to \mathcal{Z}$ is an extension then $R_\phi \subseteq R_\psi$ iff there is an extension $\eta : \mathcal{Y} \to \mathcal{Z}$ such that $\psi = \eta \circ \phi$, and η is an isomorphism (i.e., up to isomorphism $\phi = \psi$) iff $R_\phi = R_\psi$. The extension ϕ is called *weakly mixing* whenever the flow $\mathcal{R}_\phi$ is ergodic.

As in the absolute case, an extension that is both equicontinuous and weakly mixing is trivial, i.e., an isomorphism ⟦note that equicontinuity of $\phi : \mathcal{X} \to \mathcal{Z}$ can be expressed as follows: $\forall \alpha \in U_X \ \exists \beta \subseteq U_X \ : \ T\beta \cap R_\phi \subseteq \alpha$; as $T\beta \cap R_\phi = T(\beta \cap R_\phi) = R_\phi$ ($\mathcal{R}_\phi$ is ergodic), we get $\alpha \supseteq R_\phi$ for every $\alpha \in U_X$; hence $R_\phi = \Delta_X$⟧. This implies that *a weakly mixing extension has no non-trivial equicontinuous factors.* ⟦Let $\theta : \mathcal{X} \to \mathcal{Y}$ and $\psi : \mathcal{Y} \to \mathcal{Z}$ be extensions such that $\phi := \psi \circ \theta$ is weakly mixing. Then $\theta \times \theta$ maps R_ϕ onto R_ψ, hence if $\mathcal{R}_\phi$ is ergodic then $\mathcal{R}_\psi$ is ergodic, i.e., if ϕ is weakly mixing then so is its factor ψ. If ψ is also equicontinuous then ψ is an isomorphism.⟧ Again, one may ask for sufficient conditions for the converse. In order to explain the relevance of this question for the FST we mention the following results (see e.g., [9]):

THEOREM. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be an extension of flows. Then ϕ has a factorization of the form*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ \phi\ } & \mathcal{Z} \\ \phi_\infty \searrow & & \nearrow \psi_\infty \\ & \mathcal{Y}_\infty & \end{array}$$

where ϕ_∞ has no non-trivial equicontinuous factors and ψ_∞ is a (possibly transfinite) composition[4] *of equicontinuous extensions.*

PROOF (outline). If ϕ has no non-trivial equicontinuous factors then take $\phi_\infty = \phi$ and $\psi_\infty = id_Z$. In the other case, $\phi = \psi_1 \circ \phi_1$ with ψ_1 a *non-trivial* equicontinuous extension. Now apply the same reasoning to ϕ_1. Continue the process by transfinite induction, taking inverse limits for limit ordinals. We get factorizations $\phi = \psi_\lambda \circ \phi_\lambda$, where the ψ_λ are transfinite compositions of equicontinuous

[4]) We shall not give a formal definition of 'transfinite composition' of extensions. The idea will be clear from the proof.

extensions, and where the λ's run through an initial segment of the ordinals. For some λ we must have $\psi_{\lambda+1} = \psi_\lambda$: the sets $R(\psi_\lambda)$ form a decreasing family of subsets of $X \times X$, so the cardinality of the set of mutually different members of this family cannot exceed the cardinality of $X \times X$. But if $R(\psi_{\lambda+1}) = R(\psi_\lambda)$ then (up to an isomorphism) $\psi_{\lambda+1} = \psi_\lambda$. This means that ϕ_λ has no non-trivial equicontinuous factors. □

What we shall show (or at least: indicate proofs of) below are the following:

- if in the above theorem ϕ is a distal extension of minimal flows then ϕ_∞ is weakly mixing:
- a weakly mixing and distal extension of minimal flows is an isomorphism.

However, if ϕ is distal then so is ϕ_∞ ⟦clear from $R(\phi_\infty) \subseteq R_\phi$⟧, hence by these two statements ϕ_∞ is an isomorphism. This proves ([9],[7],[17]):

FST FOR DISTAL EXTENSIONS. *Every distal extension of compact minimal flows is a transfinite composition of equicontinuous extensions.*

(The technical term, often used in the literature, for a transfinite composition of equicontinuous extensions is *strictly-I-extension*, where the 'I' comes from 'isometric'.)

3 DISTAL AND NO NON-TRIVIAL EQUICONTINUOUS FACTOR IMPLIES WEAKLY MIXING

We need some preliminary results about distal extensions. For the understanding of the proofs it is sufficient that the reader is familiar with Chapter I of [13].

3.1 PROPOSITION. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be a distal extension of minimal flows. Then:*
1. The set R_ϕ is a union of minimal subsets of $\mathcal{X} \times \mathcal{X}$.
2. The mapping $\phi : X \to Z$ is open.
3. For every proximal extension $\psi : \mathcal{Y} \to \mathcal{Z}$ with $\mathcal{Y}$ a minimal flow the set

$$R(\phi, \psi) := \{(x, y) \in X \times Y \; : \; \phi(x) = \psi(y)\}$$

is minimal in the flow $\mathcal{X} \times \mathcal{Y}$ (coordinate-wise action of T on $X \times Y$).

PROOF.
1. Apply [13], II.1.2 to the (distal!) extension $(x, x') \mapsto \phi(x) : \mathcal{R}_\phi \to \mathcal{X}$.
2. See [1] ,p. 142.
3. 'Relativize' [13], II.1.3, as follows: the restriction to $R(\phi, \psi)$ of the projection $X \times Y \to X$ defines a proximal extension $\mathcal{R}(\phi, \psi) \to \mathcal{X}$, so by [13], II.1.1 the flow $\mathcal{R}(\phi, \psi)$ has a unique minimal set. Similarly, the projection $\mathcal{R}(\phi, \psi) \to \mathcal{Y}$ is a distal extension, so by [13], II 1.3, $\mathcal{R}(\phi, \psi)$ is a union of minimal sets. So $\mathcal{R}(\phi, \psi)$ is minimal. □

For the study of extensions with no non-trivial equicontinuous factors we shall

use the notion of a *continuous invariant fibre-wise pseudo-metric* (abbreviation: *CIFP*). A *CIFP* for an extension $\phi : \mathcal{X} \to \mathcal{Z}$ is a continuous mapping $\rho : R_\phi \to \mathbb{R}^+$ such that
(C.1) $\forall z \in Z : \rho\,|_{\phi^{\leftarrow}[z]\times\phi^{\leftarrow}[z]}$ is a pseudo-metric on $\phi^{\leftarrow}[z]$,
(C.2) $\forall (x_1, x_2) \in R_\phi\ \forall t \in T : \rho(tx_1, tx_2) = \rho(x_1, x_2)$.
Thus, the conditions $\rho(x_1, x_2) = \rho(x_2, x_1)$ and $\rho(x_1, x_3) \le \rho(x_1, x_2) + \rho(x_2, x_3)$ hold only for points x_1, x_2 and x_3 with $\phi(x_1) = \phi(x_2) = \phi(x_3)$. If ρ is a CIFP for ϕ on X then it is easy to see that

$$D_\phi(\rho) := \{(x_1, x_2) \in R_\phi\ :\ \rho(x_1, x_2) = 0\}$$

is a closed invariant equivalence relation on X. *If ϕ admits a CIFP ρ such that $D_\phi(\rho) = \Delta_X$ then ϕ is equicontinuous.* For let $\alpha \in U_X$. By a compactness argument there exists $\epsilon > 0$ such that $S_\epsilon := \{(x_1, x_2) \in R_\phi : \rho(x_1, x_2) \le \epsilon\} \subseteq \alpha \cap R_\phi$ $[\![\alpha$ may be assumed to be open, and $\bigcap\{S_\epsilon\ :\ \epsilon > 0\} = \Delta_X \subseteq \alpha \cap R_\phi]\!]$. There exists $\beta \in U_X$ with $\beta \cap R_\phi \subseteq S_\epsilon$ $[\![S_\epsilon$ is a nbd of Δ_X in $R_\phi]\!]$. Then $T\beta \cap R_\phi \subseteq TS_\epsilon = S_\epsilon \subseteq \alpha$. So ϕ is equicontinuous. More generally, we get:

3.2 LEMMA. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be an extension with no non-trivial equicontinuous factors. Then every CIFP ρ for ϕ is zero on R_ϕ, i.e., $D_\phi(\rho) = R_\phi$.*

PROOF. As $D_\phi(\rho)$ is a closed equivalence relation on X the quotient space $X/D_\phi(\rho)$ is a compact Hausdorff space. As $D_\phi(\rho)$ is invariant, an action of T on $X/D_\phi(\rho)$ can be unambiguously defined by $tq(x) := q(tx)$ for $t \in T$ and $x \in X$ (here $q : X \to X/D_\phi(\rho)$ is the quotient map). Thus we get a flow $\mathcal{X}/D_\phi(\rho)$. As $D_\phi(\rho) \subseteq R_\phi$ (by definition) there is an unambiguously defined continuous mapping $\psi : X/D_\phi(\rho) \to Z$ such that $\psi(qx) = \phi(x)$ for $x \in X$. Actually, $q : \mathcal{X} \to \mathcal{X}/D_\phi(\rho)$ and $\psi : \mathcal{X}/D_\phi(\rho) \to \mathcal{Z}$ are extensions of flows, and $\phi = \psi \circ q$, i.e., ψ is a factor of ϕ. Now define $\sigma : R_\psi \to \mathbb{R}^+$ by $\sigma(q(x_1), q(x_2)) := \rho(x_1, x_2)$. Then σ is a well-defined $[\![$for $R_q = D_\phi(\rho)]\!]$ CIFP for ψ on $\mathcal{X}/D_\phi(\rho)$ and it is easily seen that $D_\psi(\sigma) = \Delta_{X/D_\phi(\rho)}$. So by the remark preceding the lemma, ψ is equicontinuous. So by assumption, ψ is a bijection, which implies that $D_\phi(\rho) = R_\phi$. □

Next, we want to prove a kind of inverse to the preceding lemma: if every CIFP for ϕ is zero on R_ϕ then ϕ is weakly mixing (which, in general, is a stronger condition then 'no non-trivial equicontinuous factor'). To this end we need a device to construct CIFP's. For such a construction we need the notion of a Relatively Invariant Measure (abbreviated: RIM).

If $\mathcal{X}$ is a flow then the space of all probability measures on X will be denoted by $M_1(X)$. Recall that a probability measure on X is a regular Borel measure μ on X with $\mu(X) = 1$ ('regular' means in this case that if A is a closed set in X then $\mu(A) = \inf\{\mu(U)\ :\ U$ open and $U \supseteq A\}$). For every $\mu \in M_1(X)$ and $t \in T$ define $t\mu$ by

$$(t\mu)(A) := \mu(t^{-1}A) \text{ for every Borel set } A \text{ in } X.$$

It is easy to see that $t\mu \in M_1(X)$ for all $t \in T$ and $\mu \in M_1(X)$ and that in this way an action of T on (the set) $M_1(X)$ is defined. If $t\mu = \mu$ then μ is called an *invariant measure* on $\mathcal{X}$. The 'relativization' of this notion is as follows:

An extension $\phi : \mathcal{X} \to \mathcal{Z}$ of flows is called a *RIM-extension* (or: is said *to have a RIM*) whenever there exists a mapping $\lambda : Z \to M_1(X)$ such that (writing λ_z for $\lambda(z)$):
(R.1) $\forall f \in C(X) : z \mapsto \int_X f d\lambda_z : Z \to \mathbb{R}$ is continuous,
(R.2) $\forall (t, z) \in T \times Z : \lambda_{tz} = t\lambda_z$ ('invariance'),
(R.3) $\forall z \in Z :$ Supp $\lambda_z \subseteq \phi^{\leftarrow}[z]$.
Condition (R.3) is equivalent to saying that $\lambda_z(X \setminus \phi^{\leftarrow}[z]) = 0 : \lambda_z$ 'lives' in the fiber $\phi^{\leftarrow}[z]$. (Note that an extension of the form $\mathcal{X} \to (*)$ has a RIM iff $\mathcal{X}$ has an invariant measure.) A mapping λ satisfying the conditions (R.1), (R.2) and (R.3) will be called a *section* for ϕ. For the existence of RIM-extensions, see 3.7 below.

Consider a RIM-extension $\phi : \mathcal{X} \to \mathcal{Z}$ with section $\lambda : Z \to M_1(X)$ and let N be a closed invariant set in $\mathcal{R}_\phi$. For $x \in X$, put

$$N[x] := \{x' \in X \; : \; (x, x') \in N\}.$$

Clearly, $N[x]$ is closed in $X, N[x] \subseteq \phi^{\leftarrow}[x]$ and $tN[x] = N[tx]$ for all $t \in T$. Moreover, if $\mathcal{X}$ is minimal and $N \neq \emptyset$ then $N[x] \neq \emptyset$ for all $x \in X$ [the projection of $X \times X$ onto its first factor X is an extension of flows $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$, hence it maps N *onto* X]. For an arbitrary non-empty closed invariant set N in $\mathcal{R}_\phi$ put

$$\rho_N(x_1, x_2) := \lambda_{\phi(x_1)}(N[x_1] \triangle N[x_2]) \quad \text{for } (x_1, x_2) \in R_\phi.$$

Here '$\triangle$' denotes the symmetric difference of sets. Note that ρ_N is well defined because the symmetric difference of the closed sets $N[x_1]$ and $N[x_2]$ is a Borel set in X. Now we have:

3.3 LEMMA. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be a RIM-extension of minimal flows with section λ and let N be a non-empty closed invariant set in $\mathcal{R}_\phi$. Then ρ_N as defined above is a CIFP.*

PROOF. It is straightforward to check the conditions (C.1) and (C.2) for ρ_N (for the proofs one needs only (R.2)). The proof that ρ_N is continuous is a bit tricky and uses condition (R.1) and regularity of all involved measures as well as minimality of $\mathcal{X}$. We shall not give it here; for details, see [20] (or 'relativize' the proof on p. 128/129 of [1]). □

The following 'technical' lemma is precisely the heart of the reason that RIM's are useful.

3.4 LEMMA. *Let ϕ, λ and ρ_N be as above. Then for every $(x_1, x_2) \in D_\phi(\rho_N)$ and every non-empty open set U in X:*

$$U \cap \text{Supp}\, \lambda_{\phi(x_2)} \subseteq N[x_2] \Rightarrow U \cap \text{Supp}\, \lambda_{\phi(x_2)} \subseteq N[x_1].$$

PROOF. Assume the left-hand inclusion. Then

$$\lambda_{\phi(x_2)}(U \setminus N[x_1]) \leq \lambda_{\phi(x_2)}(N[x_2] \setminus N[x_1]) \leq \rho_N(x_2, x_1) = 0.$$

As $U \setminus N[x_1]$ is an open set this means that $U \setminus N[x_1]$ is disjoint from Supp $\lambda_{\phi(x_2)}$. So $U \cap$ Supp $\lambda_{\phi(x_2)} \subseteq N[x_1]$. □

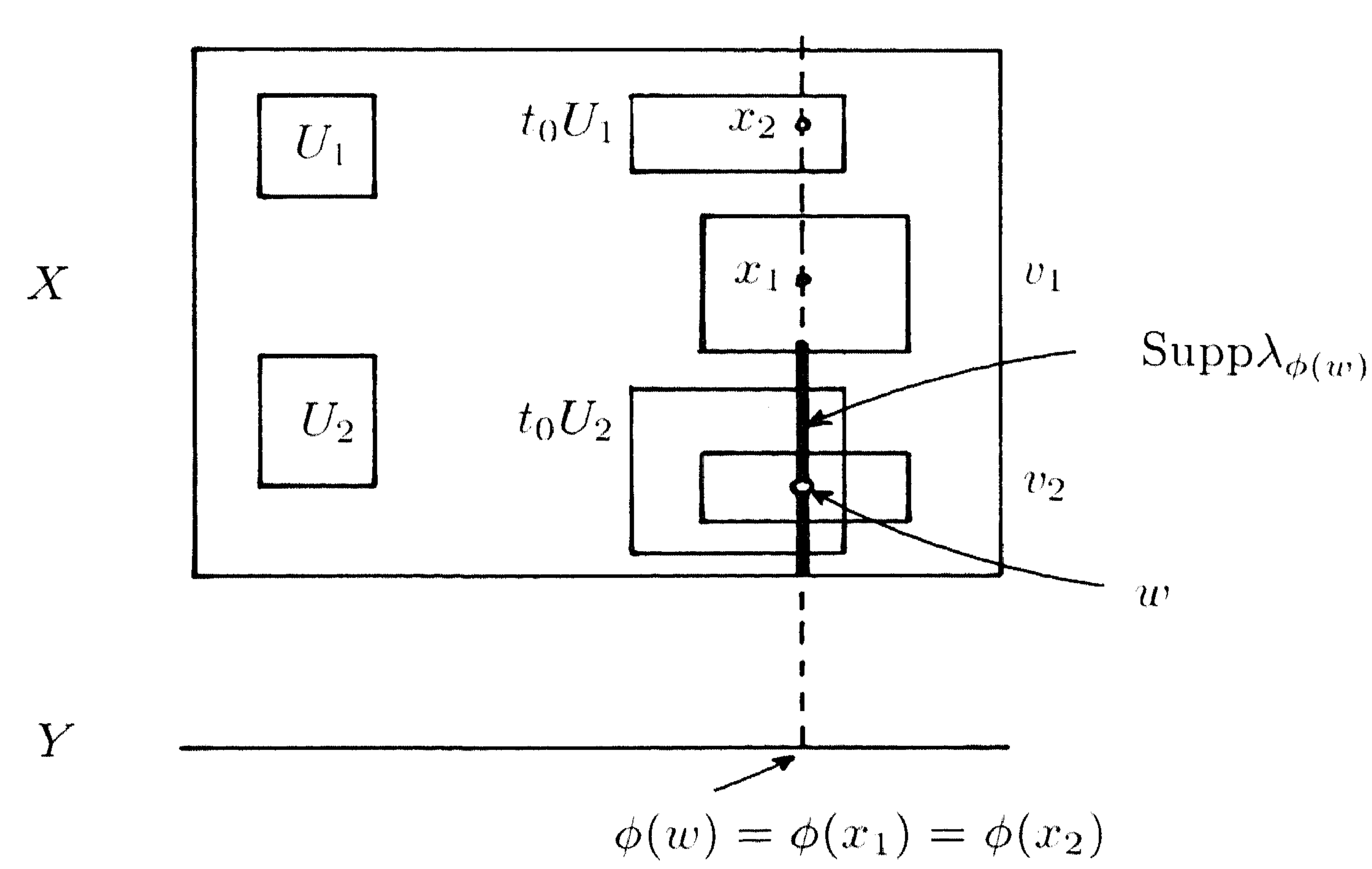
3.5 LEMMA. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be an open RIM-extension of minimal flows and assume that every CIFP for ϕ is zero on R_ϕ. Then ϕ is weakly mixing.*

PROOF. We have to show that if $(U_1 \times U_2) \cap R_\phi$ and $(V_1 \times V_2) \cap R_\phi$ are non-empty basic open sets in R_ϕ (U_i and V_i open in X for $i = 1, 2$) then $(V_1 \times V_2) \cap T(U_1 \times U_2) \cap R_\phi \neq \emptyset$ or, equivalently, $(V_1 \times V_2) \cap N \neq \emptyset$, where $N := \overline{T(U_1 \times U_2) \cap R_\phi}$. Note that N is a closed invariant subset of R_ϕ, so according to 3.3 we have a CIFP ρ_N for ϕ. By assumption, this CIFP is zero on R_ϕ.

Observe that we may assume that $\phi[U_1] = \phi[U_2]$ and $\phi[V_1] = \phi[V_2]$ [[replace U_i by $U_i \cap \phi^{\leftarrow}[U]$, where $U := \phi[U_1] \cap \phi[U_2] \neq \emptyset$ and open (because ϕ is open); similar for V_i ($i = 1, 2$)]]. Let $t_0 \in T$ with $W := t_0 U_2 \cap V_2 \neq \emptyset$ [[such t_0 exists because $\mathcal{X}$ is minimal]] and select $w \in W$ such that $w \in \mathrm{Supp}\lambda_{\phi(w)}$. [[This is possible: pick any $x_0 \in X$ and $x' \in \mathrm{Supp}\,\lambda_{\phi(x_0)} \subseteq \phi^{\leftarrow}[\phi x_0]$. Then $\phi(x') = \phi(x_0)$, so $x' \in \mathrm{Supp}\,\lambda_{\phi(x')}$, hence $tx' \in \mathrm{Supp}\,\lambda_{\phi(tx')}$ for all $t \in T$. As x' has a dense orbit one can take t such that $w := tx' \in W$.]] By the choice of U_i and V_i ($i = 1, 2$) there are $x_1 \in V_1$ and $x_2 \in t_0 U_1$ such that $\phi(x_1) = \phi(w) = \phi(x_2)$. Then

$$\emptyset \neq (\{x_2\} \times W) \cap R_\phi \subseteq t_0(U_1 \times U_2) \cap R_\phi \subseteq N,$$

and it follows that $W \cap \phi^{\leftarrow}[\phi x_2] \subseteq N[x_2]$, hence certainly $W \cap \mathrm{Supp}\,\lambda_{\phi(x_2)} \subseteq N[x_2]$. Now recall that ρ_N is zero on R_ϕ, so that in particular $\rho_N(x_1, x_2) = 0$. So by 3.4 we get $W \cap \mathrm{Supp}\,\lambda_{\phi(x_2)} \subseteq N[x_1]$. As $w \in W$ and $w \in \mathrm{Supp}\,\lambda_{\phi(w)} = \mathrm{Supp}\,\lambda_{\phi(x_2)}$ [[$\phi(x_2) = \phi(w)$]] it follows that $w \in N[x_1]$, that is, $(x_1, w) \in N$. But also $(x_1, w) \in V_1 \times V_2$, so $(V_1 \times V_2) \cap N \neq \emptyset$. □



3.6 COROLLARY. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be an open RIM-extension of minimal flows. If ϕ has no non-trivial equicontinuous factors then ϕ is weakly mixing.*

PROOF. Clear from 3.2 and 3.5. □

We want to replace in the above Corollary the condition 'open RIM' by 'distal'. (It can be shown that every distal extension of minimal flows is an open RIM-extension. But for the proof - at least, as far as I know - one needs the general form of FST we want to prove.) For that purpose we shall use the following theorem.

3.7 THEOREM [12]. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be an extension of minimal flows. Then there exists a commutative diagram of extensions of minimal flows*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\sigma} & \mathcal{X} \\ \phi' \downarrow & & \downarrow \phi \\ \mathcal{Z}' & \xrightarrow{\tau} & \mathcal{Z} \end{array}$$

with the following properties:
(a) σ and τ are proximal extensions,
(b) ϕ' is a RIM-extension,
(c) X' is a minimal set in $\mathcal{R}(\phi, \tau)$[5] *and σ and ϕ' are the restriction to X' of the canonical projections of $X \times Z'$ onto X and Z', respectively.*

PROOF. See [12]. The proof uses methods from functional analysis, like the Krein-Milman theorem and properties of the 'barycenter map' of compact convex sets in locally convex spaces. The flow $\mathcal{Z}'$ is obtained as a certain subflow of the natural flow on $M_1(X)$. □

3.8 LEMMA. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be a distal extension of minimal flows. Then in the diagram of 3.7 we have:*
1. ϕ' is distal, hence open.
2. $X' = R(\phi, \tau)$, hence $(\sigma \times \sigma)[R_{\phi'}] = R_\phi$.

PROOF.
1. Let $(x'_1, x'_2) \in R_{\phi'}$. Then clearly $(\sigma x'_1, \sigma x'_2) \in R_\phi$, so $(\sigma x'_1, \sigma x'_2)$ is a distal pair (by the definition of 'distal extension'). If (x'_1, x'_2) were a proximal pair then also $(\sigma x'_1, \sigma x'_2)$ would be a proximal pair ⟦straightforward⟧, so we would have $\sigma x'_1 = \sigma x'_2$. Together with $\phi'(x'_1) = \phi'(x'_2)$ this would imply $x'_1 = x'_2$ ⟦ϕ and σ are projections⟧. Thus, (x'_1, x'_2) is either not proximal, or $x'_1 = x'_2$: so by definition (x'_1, x'_2) is a distal pair. This shows that ϕ' is a distal extension. By 3.1.2, ϕ' is open.
2. By 3.1.3, $\mathcal{R}(\phi, \tau)$ is minimal, hence $X' = R(\phi, \tau)$. Now consider $(x_1, x_2) \in R_\phi$ and let $y := \phi(x_1) = \phi(x_2)$ and $y' \in \tau^{\leftarrow}[y]$. Then for $i = 1, 2$ we have $(x_i, y') \in R(\phi, \tau) = X'$, and therefore $((x_1, y'), (x_2, y')) \in R_{\phi'}$. This element of $R_{\phi'}$ is mapped onto (x_1, x_2) by $\sigma \times \sigma$. This shows that $(\sigma \times \sigma)[R_{\phi'}] \supseteq R_\phi$. The

[5]) For the definition of $R(\phi, \tau)$, see 3.1.3. By $\mathcal{R}(\phi, \tau)$ we denote the subflow of $\mathcal{X} \times \mathcal{Z}'$ on $R(\phi, \tau)$.

inclusion $(\sigma \times \sigma)[R_{\phi'}] \subseteq R_\phi$ is trivial. □

3.9 PROPOSITION. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be a distal extension of minimal flows. If ϕ has no non-trivial equicontinuous factors then ϕ is weakly mixing.*

PROOF. Consider a diagram as in Theorem 3.7. We shall first show that ϕ' is weakly mixing. By 3.7 (b) and 3.8.1, ϕ' is an open RIM-extension, so by 3.6 it is sufficient to show that every CIFP for ϕ' is zero on $R_{\phi'}$. So let ρ' be a CIFP for $R_{\phi'}$. Claim: ρ' can be factorized as $\rho' = \rho \circ (\sigma \times \sigma)$ with ρ a CIFP for ϕ. Assuming that this has been proved, we proceed as follows: by 3.2 and the hypothesis on ϕ, ρ is zero on R_ϕ, hence ρ' is zero on $R_{\phi'}$ just what we wanted to prove. This completes the proof that ϕ' is weakly mixing, i.e., that $\mathcal{R}_\phi$ is an ergodic flow. As $\sigma \times \sigma$ maps $R_{\phi'}$ onto R_ϕ, it follows easily that $\mathcal{R}_\phi$ is an ergodic flow, i.e., that ϕ is a weakly mixing extension. This completes the proof of the proposition. It remains to prove the above claim: every CIFP ρ' for ϕ' factorizes as $\rho' = \rho \circ (\sigma \times \sigma)$ with ρ a CIFP for ϕ. The proof is as follows.

Let $(x_1, x_2) \in R_\phi$. Then there exists $(x'_1, x'_2) \in R_{\phi'}$ with $(\sigma \times \sigma)(x'_1, x'_2) = (x_1, x_2)$ ⟦cf. 3.8.2⟧; put $\rho(x_1, x_2) := \rho'(x'_1, x'_2)$. This unambiguously defines a function $\rho : R_\phi \to \mathbb{R}^+$. For if also $(\sigma \times \sigma)\ (x''_1, x''_2) = (x_1, x_2)$ for $(x''_1, x''_2) \in R_{\phi'}$, then there are z', $z'' \in Z'$ such that $x'_i = (x_i, z')$ and $x''_i = (x_i, z'')$ for $i = 1, 2$ ⟦recall that $x' = (\sigma(x'), \phi'(x'))$ for every $x' \in X'$; so just take $z' = \phi'(x'_1) = \phi'(x'_2)$ and $z'' = \phi'(x''_1) = \phi'(x''_2)$⟧. Note that $\tau(z') = \tau(z'')$ ⟦$\phi \circ \sigma = \tau \circ \phi'$ and $\sigma(x'_i) = \sigma(x''_i)$⟧, hence (z', z'') is a proximal pair. It follows that there is a net $\{t_\lambda\}_{\lambda \in \Lambda}$ in T such that the nets $\{t_\lambda z'\}_\lambda$ and $\{t_\lambda z''\}_\lambda$ converge to the same point $z^0 \in Z'$. Along a suitable subnet we have $t_\lambda x_i \rightsquigarrow \overline{x}_i$ for some $\overline{x}_i \in X$ ⟦X is compact⟧, so if we put $x^0_i := (\overline{x}_i, z^0)$ for $i = 1, 2$, then

$$t_\lambda x'_i = (t_\lambda x_i, t_\lambda z') \rightsquigarrow (\overline{x}_i, z^0) = x^0_i \text{ for } 1 = 1, 2$$

along this subnet (the same subnet for x'_1 and x'_2); similarly, $t_\lambda x''_i \rightsquigarrow x^0_i$ for $i = 1, 2$ along this subnet. It follows that

$$\rho'(x'_1, x'_2) = \rho'(t_\lambda x'_1, t_\lambda x'_2) \rightsquigarrow \rho'(x^0_1, x^0_2)$$

⟦the equality follows from condition (C.2) for a CIFP⟧, hence $\rho'(x'_1, x'_2) = \rho'(x^0_1, x^0_2)$. Similarly, $\rho'(x''_1, x''_2) = \rho'(x^0_1, x^0_2)$, and we may conclude that $\rho'(x'_1, x'_2) = \rho'(x''_1, x''_2)$. This shows that in the above the function ρ is unambiguously defined. Note that by the definition of ρ we have $\rho' = \rho \circ (\sigma \times \sigma)$. As $\sigma \times \sigma$ is a quotient map it follows that ρ is continuous, and the conditions (C.1) and (C.2) are easily verified for ρ. This completes the proof of the claim. □

4 'DISTAL AND WEAKLY MIXING' IMPLIES 'TRIVIAL'

In the 'absolute case' we have seen that an ergodic and distal flow is minimal, and that in the metric case this was quite easy to prove. The corresponding 'relative' result is:

4.1 THEOREM. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be an extension of minimal flows. If ϕ is both distal and weakly mixing then ϕ is an isomorphism.*

PROOF. In the case that X and Z are metric spaces the proof is easy: in that case the ergodic flow $\mathcal{R}_\phi$ contains a point with dense orbit. But by 3.1.1, R_ϕ is a union of minimal sets. Hence $\mathcal{R}_\phi$ is a minimal flow: the point with dense orbit is situated in a minimal set M of $\mathcal{R}_\phi$, so M is dense in R_ϕ, i.e., $M = R_\phi$. But Δ_X is a closed invariant set in $\mathcal{R}_\phi$, so $\Delta_X = R_\phi$. This implies that ϕ is injective, hence ϕ is an isomorphism.

To reduce the general case to the metric case a construction has to be used that, for the absolute case, was invented by R. Ellis in [7], and that was 'relativized' by McMahon & Wu in [17]. For any *open* extension (and recall that a distal extension is open: see 3.1.2) $\phi : \mathcal{X} \to \mathcal{Z}$, for any countable subgroup H of T and for any continuous pseudo-metric d on X there exists a commutative diagram of continuous surjections

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Z \\ \sigma_H \downarrow & & \downarrow \tau_H \\ X_H^* & \xrightarrow{\phi_H} & Z_H \end{array}$$

with the following properties:

(a) X_H^* and Z_H are compact *metric* spaces.

(b) There exist continuous actions of H on X_H^* and Z_H such that σ_H, ϕ_H and τ_H are morphisms of H-flows (note that by restricting the action of T to H also ϕ is a morphism of H-flows).

We shall not describe the details of the construction. In [17] it is shown that if ϕ is not an isomorphism then the continuous pseudo-metric d on X can be chosen such that ϕ_H is not an isomorphism. Fix d such that this is the case. On the other hand, if $\phi : \mathcal{X} \to \mathcal{Z}$ is a weakly mixing extension of minimal flows (for T) then (for given d) H can be chosen such that ϕ_H is a weakly mixing extension of minimal H-flows. Moreover, it is not so difficult (using standard techniques which we have not discussed) to show that if ϕ is distal then ϕ_H is distal as well (provided X_H^* is minimal under H). Conclusion: if ϕ is a distal and weakly mixing extension of minimal T-flows, then for suitable H, ϕ_H is a distal and weakly mixing extension of *metric* minimal H-flows. Hence ϕ_H is an isomorphism: contradiction. This implies that ϕ is an isomorphism. □

REMARK. The observation that, for suitable H, ϕ_H is weakly mixing and distal is not in [17]: in that paper ϕ_H is only shown to be 'point-distal' and to have no equicontinuous factors. Then a quite deep result from [6] is invoqued to show that ϕ_H is an isomorphism. A complete version of the above proof will be published in the near future.

4.2 COROLLARY. *Let $\phi : \mathcal{X} \to \mathcal{Z}$ be a distal extension of minimal flows. If ϕ has no non-trivial equicontinuous factors then ϕ is an isomorphism.*

PROOF. Clear from 3.9 and 4.1. □

As is observed at the end of Section 2, this is just what is needed to complete the proof of FST.

4.3 REMARKS (for specialists only).
1. The proof of 3.5 is a simplification of arguments used in [20]. These, in turn, were based on techniques from [16].
2. By a minor modification the proof of 3.5 can be adapted to a proof of the following: Let $\phi : \mathcal{X} \to \mathcal{Z}$ be an open RIM-extension which has no non-trivial equicontinuous factors. Then for every $x_0 \in X$ and every non-empty relatively open subset O of Supp $\lambda_{\phi(x_0)}$ the set $T(\{x_0\} \times O)$ is dense is R_ϕ. By the same method of proof as used on p. 221 of [1] (due to S. Glasner) it then follows that if X is metrizable every fiber $\phi^{\leftarrow}[\phi x_0]$ of ϕ has a dense subset of points x such that (x_0, x) has a dense orbit in $\mathcal{R}_\phi$. Using this it is standard to show that an open RIM-extension of metric minimal flows that is point-distal and has no equicontinuous factors is an isomorphism. Using an 'improved' version of 3.7 (with (b) replaced by: ϕ' is an open RIM-extension; see [22]) one then concludes: *a point-distal extension of metric minimal flows with no non-trivial equicontinuous factors is an isomorphism.* Thus, also the main result of [6] can be proved without using 'τ-topologies' via RIM's.
3. The proof of 3.9 is based on a technique that is used in [11].
4. The techniques used in Section 4 can be used in precisely the same way to show that a RIC-extension of compact minimal flows which has no non-trivial equicontinuous factors is weakly mixing. Using the 'improved' version of 3.7 one obtains a similar conclusion for Bronshtejn extensions (cf. [19]).
5. Using the techniques of Section 3 it is easy to see that if $\phi : \mathcal{X} \to \mathcal{Y}$ is an equicontinuous extension of minimal flows then $\bigcap\{D_\phi(\rho) : \rho$ is a CIFP for $\phi\} = \Delta_X$. If X is metrizable then in this intersection one needs only countably many CIFP's, say, $\rho_1, \rho_2, \ldots$ Then $\rho_0 := \Sigma_{i=1}^{\infty} 2^{-i}\rho_i$ is a CIFP for ϕ and $D_\phi(\rho_0) = \Delta_X$. It follows ρ_0 defines a compatible metric on each fiber. So in this case, ϕ is an isometric extension according to the definition in Section 1.

REFERENCES

1. J. AUSLANDER, *Minimal flows and their extensions*, Amsterdam, 1988.
2. J. AUSLANDER, L. GREEN & F. HAHN, *Flows on homogeneous spaces*, Princeton, 1963.
3. I.U. BRONSTEIN, *Extensions of minimal transformation groups*, Alphen aan den Rijn, 1979.
4. R. ELLIS, Locally compact transformation groups, *Duke Math. J.* **24** (1957), 119-125.
5. R. ELLIS, *Lectures on topological dynamics*, New York, 1969.
6. R. ELLIS, The Veech structure theorem, *Trans. Amer. Math. Soc.* **186** (1973), 203-218.
7. R. ELLIS, The Furstenberg structure theorem, *Pacific J. Math.* **76** (1978), 345-349.
8. R. ELLIS, S. GLASNER & L. SHAPIRO, Proximal-isometric (PI) flows, *Advances in Math.* **17** (1975), 213-260.

9. H. FURSTENBERG, The structure of distal flows, *Amer. J. Math.* **85** (1963), 477-515.
10. H. FURSTENBERG, *Recurrence in ergodic theory and combinatorial number theory*, Princeton, 1981.
11. H. FURSTENBERG & S. GLASNER, On the existence of isometric extensions, *Amer. J. Math.* **100** (1978), 1185-1212.
12. S. GLASNER, Relatively invariant measures, *Pacific J. Math.* **58** (1975), 393-410.
13. S. GLASNER, *Proximal flows*, Berlin, 1976.
14. W.H. GOTTSCHALK & G.H. HEDLUND, *Topological dynamics*, Providence, 1955.
15. E. HEWITT & K.A. ROSS, *Abstract harmonic analysis, I*, Berlin, 1963.
16. D.C. MCMAHON, Relativized weak disjointness and relatively invariant measures, *Trans. Amer. Math. Soc.* **236** (1978), 225-237.
17. D.C. MCMAHON & T.S. WU, Distal homomorphisms of non-metric minimal flows, *Proc. Amer. Math. Soc.* **82** (1981), 283-287.
18. V.V. NEMYTSKIĬ & V.V. STEPANOV, *Qualitative theory of differential equations*, Princeton, 1960.
19. W.A. VEECH, Topological dynamics, *Bull. Amer. Math. Soc.* **83** (1977), 775-830.
20. J. DE VRIES & J.C.S.P. VAN DER WOUDE, Invariant measures and the equicontinuous structure relation, II: the relative case, *Suppl. ai Rend. Circ. Mat. Palermo. Serie II. no 6* (1984), 291-306.
21. A. WILANSKY, *Topology for analysis*, Malabar, 1983.
22. J.C.S.P. VAN DER WOUDE, *Topological Dynamix*, Amsterdam, 1986.